Lumped-Element Modeling

- Last lecture
  - Two-port element example: inductor
  - Transducers
    - Classification
    - Linear, Conservative
    - General Two-Port Theory

- Today:
  - Review of Electromagnetics
  - Electrodynamic transduction
Maxwell's Equations

Differential Form

1. $\vec{\nabla} \cdot \vec{D} = \rho$
2. $\vec{\nabla} \cdot \vec{B} = 0$
3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
4. $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

Integral Form

1. $\int_S \vec{D} \cdot \hat{n} dS = q$
2. $\int_S \vec{B} \cdot \hat{n} dS = 0$
3. $-\oint_c \vec{E} \cdot d\vec{l} = \int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = \frac{d\Phi}{dt}$
4. $\oint_c \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$

Gauss's Law
Gauss's Law
Faraday's Law
Ampere's Law
Review of Electromagnetics

Constitutive Relations

\[ \vec{J} = \sigma \vec{E} \quad \text{Ohm's Law} \]
\[ \vec{D} = \varepsilon \vec{E} \quad \text{Permitivity} \]
\[ \vec{B} = \mu \vec{H} \quad \text{Permeability} \quad \text{where } \mu = \mu_r \mu_0 \text{ and } \mu_0 = 4\pi \times 10^{-7} \frac{H}{m} \]

Continuity Equation

\[ \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \]

Lorentz Force

\[ \vec{F} = \vec{F}_{\text{electric}} + \vec{F}_{\text{magnetic}} \]
\[ = q \left( \vec{E} + \vec{u} \times \vec{B} \right) \]
Magnetic Transduction

- Magnetic Transduction
  Motor/generator action are produced by variations of the attractive force tending to close the air gap in a ferromagnetic circuit.

- Fundamental Definitions
  - Magnetic field, $H$ (Units: $A/m$)
  - Scalar magnetic potential, $M$ (Units: $A$) Also known as Magneto Motive Force (MMF)
    
    \[ F_{MM} = \int_{x_1}^{x_2} \vec{H} \cdot d\vec{l} \]

  - Magnetic flux, $\phi$ (Units: Weber=AH=V-sec)
  - Magnetic flux density, $B$ (Units: $Wb/m^2$)
    
    \[ \phi = \int_S \vec{B} \cdot \hat{n} dS \]
Analogy

**Magnetic quantities**

**Electrical quantities**

**MMF is effort.**

\[ F = HI \]

\[ V = EI \]

\[ \phi \text{ is displacement. } \frac{d\phi}{dt} \text{ is flow. MMF and } \frac{d\phi}{dt} \text{ are conjugate power variables (Check!). No exact analog of } q \text{ for magnetics.} \]

Ref. R. W. Erickson, Fundamentals of Power Electronics, p. 456
Faraday's Law (integral form) relates voltage, $v(t)$, induced in a loop of wire to the time derivative of the total flux passing through the winding:

$$v(t) = -\frac{d\phi(t)}{dt}$$

('sign' is given by Lenz's law.)

Ampere's Law (integral form) relates magnetomotive force, $M(t)$, induced in magnetic core to the total current passing through interior of path:

$$F_{MM}(t) = \oint_C \vec{H}(t) \cdot d\vec{l} = H(t)l_m = i(t)$$

if magnetic field is uniform.

Ref. R. W. Erickson, Fundamentals of Power Electronics, p. 457-458
Magnetic Circuits

Consider a magnetic element of length, \( \ell \), and area, \( A_c \). Given a uniform magnetic field, \( H \), over the length, \( \ell \), the induced MMF (scalar magnetic potential) is:

\[
F_{MM} = H\ell = \left( \frac{B}{\mu} \right) \ell \quad \text{[using } B = \mu H \text{]}
\]

\[
F_{MM} = \left( \frac{\phi}{A_c} \right) \ell \quad \text{[using } \phi = BA_c \text{ assuming uniform magnetic flux density]}
\]

\[
F_{MM} = \left( \frac{\ell}{\mu A_c} \right) \phi = R\phi \quad \text{[R is called reluctance (not to be confused with resistance!)]}
\]

Kirchoff-like Laws apply.

(1) Divergence of magnetic flux is zero at a node. \( \vec{V} \cdot \vec{B} = 0 \) indicates total flux entering node must be zero.

(2) KML (Kirchoff's Magnetomotive Force Law)

Sum of MMF: \( \int_C \vec{H}(t) \cdot d\vec{l} = ni(t) \)

where \( n \) = # of turns of wire carrying current \( i \).

Electromagnetic Transduction

Since $\frac{d\phi}{dt}$ is flow, $\phi$ displacement. From $\phi = \frac{1}{R} F_{MM}$, what is the reluctance???

The magnetic reluctance, $R$, is analogous to the spring constant.

In the magnetic energy domain, the magnetic element stores potential energy.
In the electrical energy domain, the MMF is related to the electrical current by $F_{MM} = ni$, and the inductor stores kinetic energy in the current flow.

The coupling equations between the electrical domain and magnetic domain are:

$$\dot{\phi} = \frac{v}{n} \quad \& \quad F_{MM} = ni \quad \text{where} \quad \phi = \frac{1}{R} F_{MM}$$

From these, we can calculate the electrical inductance:

$$v(t) = L \frac{di}{dt} \quad \text{where} \quad L = \frac{n^2}{R}.$$
Electromagnetic Transduction

- Circuit Representation:
  - Transformer: impedance analogy to admittance analogy

\[
\begin{bmatrix}
\dot{\phi} \\
F_{MM}
\end{bmatrix} =
\begin{bmatrix}
0 & 1/n \\
n & 0
\end{bmatrix} \begin{bmatrix}
i \\
V
\end{bmatrix}
\]

\[\begin{array}{c}
V \\
i
\end{array}\quad 1:n \quad \begin{array}{c}
- \\
\phi
\end{array}\]

\[\begin{array}{c}
T \ (Y \text{ to } Z) \\
n = \# \text{ wire turns}
\end{array}\]

- Gyrator: impedance analogy to impedance analogy

\[
\begin{bmatrix}
\dot{\phi} \\
F_{MM}
\end{bmatrix} =
\begin{bmatrix}
0 & 1/n \\
n & 0
\end{bmatrix} \begin{bmatrix}
i \\
V
\end{bmatrix}
\]

\[\begin{array}{c}
i \\
\phi
\end{array}\quad \begin{array}{c}
- \\
F_{MM}
\end{array}\quad \begin{array}{c}
G \ (Z \text{ to } Z)
\end{array}\]
Example: Inductor with Air Gap

KML: \( M_{\text{core}} + M_{\text{gap}} = ni \) and continuity of flux:

\[
\begin{align*}
\phi_{\text{core}} &= \phi_{\text{gap}} = \phi \\
R_{\text{core}}\phi_{\text{core}} + R_{\text{gap}}\phi_{\text{gap}} &= ni \\
\phi(R_{\text{core}} + R_{\text{gap}}) &= ni
\end{align*}
\]

Faraday's Law for n turns: \( \nu(t) = n\frac{d\phi}{dt} \)

\[
\nu(t) = \left( \frac{n^2}{R_{\text{core}} + R_{\text{gap}}} \right) \frac{di}{dt}
\]

\[\Rightarrow L = \left( \frac{n^2}{R_{\text{core}} + R_{\text{gap}}} \right) \text{ where}\]

\[
R_{\text{core}} = \frac{1}{\mu A_c} \text{ and } R_{\text{gap}} = \frac{g}{\mu_0 A_c}
\]


Caution: \( \phi \) is not a conjugate power variable!
The stored potential energy in the inductor (magnetic energy domain) is:

\[ dW_{PE} = edq = F_{MM} d\phi \]

\[ W_{PE} = \frac{\phi^2 R}{2} \]

\[ dW_{PE}^* = qde = \phi dF_{MM} \]

\[ W_{PE}^* = \frac{F_{MM}^2}{2R} \]

Since \( F_{MM} = R\phi \),

Therefore, \( W_{PE} = W_{PE}^* \)

Also, \( F_{MM} = ni \),

So, \( W_{PE}^* = \frac{1}{2} Li^2 \)
Electrodynamic Transduction

- **Electrodynamic**: motor/generator action are produced by the current in, or the motion of an electric conductor located in a fixed transverse magnetic field (i.e., voice coil, solenoid, etc.).

  ![Diagram](image)

  - Lenz's law: "relates motion of conductor in a magnetic field to the induced open-circuit voltage across terminals 1-2". For a differential element, \( dV = (\vec{u} \times \vec{\beta}) \cdot d\vec{l} \), where \( \vec{\beta} = \) magnetic flux density.

  - Total induced voltage is \( \int dV = V = \beta \ell u \).

  Velocity is "upward".

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Electrodynamic Transduction

- **Electrodynamic**: motor/generator action are produced by the current in, or the motion of an electric conductor located in a fixed transverse magnetic field (i.e., voice coil, solenoid, etc.).

Laplace's law: "relates force on a conductor in a magnetic field to the current passing through the conductor". For a differential element, $d\vec{F}_{mag} = id\vec{\ell} \times \vec{\beta}$. The total induced force is $\int d\vec{F}_{mag} = F_{mag} = \beta \ell i$. Force is "upward".

**"Left-hand rule"**

Electrodynamic Transduction

**Characteristic Transducer Equations:**

From Lenz's and Laplace's laws we get the characteristic questions:

\[ V = \beta \ell u \quad \text{and} \quad F_{\text{mag}} = \beta \ell i \]

or in matrix form,

\[
\begin{bmatrix}
V \\
F_{\text{mag}}
\end{bmatrix} =
\begin{bmatrix}
0 & T_{EM} \\
T_{ME} & 0
\end{bmatrix}
\begin{bmatrix}
i \\
u
\end{bmatrix}, \text{ where } T_{EM} = T_{ME} = \beta \ell
\]

Note: \( Z_{EB} = Z_{MO} = 0 \), so there is direct coupling between \( V \) and \( u \) or \( F_{\text{mag}} \) and \( i \).
Electrodynamic Transduction

- **Circuit Representation:**
  - Transformer: impedance analogy to admittance analogy
    
    \[
    \begin{bmatrix}
    U \\
    F_{mag}
    \end{bmatrix} = \begin{bmatrix}
    1/\beta l & 0 \\
    0 & \beta l
    \end{bmatrix}
    \begin{bmatrix}
    V \\
    I
    \end{bmatrix}
    \]

    \[
    n = \frac{1}{\beta \ell}
    \]
    
    T (Z to Y)

  - Gyrator: impedance analogy to impedance analogy
    
    \[
    \begin{bmatrix}
    F_{mag} \\
    U
    \end{bmatrix} = \begin{bmatrix}
    0 & \beta l \\
    1/\beta l & 0
    \end{bmatrix}
    \begin{bmatrix}
    V \\
    I
    \end{bmatrix}
    \]

    \[
    n = \beta \ell
    \]
    
    G (Z to Z)
Example: Loud Speaker


Admittance ↔ Impedance