Lumped-Element Modeling

- Last lecture
  - Single-port element examples
  - Two-port elements
    - Transformer
    - Gyrator

- Today:
  - Two-port element example: Inductor
  - Transducers
    - Classification
    - Linear, Conservative
    - General Two-Port Theory

- Reading: Senturia, pp. 120-123, 142-145.
Impedance Transformations

\[
Z(s) = \frac{e_2}{f_2} = \frac{ne_1}{f_1} = n^2 \cdot \frac{e_1}{f_1}
\]

\[
Z_{\text{in}}(S) = \frac{e_1}{f_1} = \frac{Z(s)}{n^2}
\]

Impedance Scaling.

\[
Z(s) = \frac{e_2}{f_2} = \frac{nf_1}{e_1} = n^2 \cdot \frac{f_1}{e_1}
\]

\[
Z_{\text{in}}(S) = \frac{e_1}{f_1} = \frac{n^2}{Z(s)}
\]

Impedance \rightarrow \text{Admittance.}
Capacitor \rightarrow \text{Inductor}
Lever/Transformer:

Balancing the moments requires, $F_1 \cdot L_1 = F_2 \cdot L_2$ or if not balanced $\omega = \frac{u_1}{L_1} = \frac{u_2}{L_2}$. So the transformer law requires $n = \frac{L_1}{L_2}$.
Two-Port Element Example

**Electrical Inductor:**

Magnetomotive force (MMF) is the strength of a magnetic field in a coil of wire, depending on the current and the number of turns of the coil.

\[ F_{MM} = nI \quad (1) \]

- **n**: number of turns
- **I**: current

Reluctance \( R \)

\[ F_{MM} = R \phi \quad (2) \]

- **\( \phi \)**: magnetic flux

- **Faraday’s Law of Induction**

\[ \frac{\dot{\phi}}{n} = \frac{1}{V} \quad (3) \]

- **V**: induced voltage, EMF
Two-Port Element Example

**Electrical Inductor:**

- Equations (1) and (3) can be rewritten as the following:

\[
\begin{bmatrix}
\dot{\phi} \\
F_{MM}
\end{bmatrix} = \begin{bmatrix}
0 & 1/n \\
n & 0
\end{bmatrix} \begin{bmatrix}
I \\
V
\end{bmatrix}
\]

- \(Z_m\) is readily given by

\[
Z_m = \frac{F_{MM}}{s\phi}
\]

- Considering Equation (2), we have

\[
Z_m = \frac{1}{s(1/R)}
\]

Therefore, a gyrator can be used.

Diagram:

- Capacitor
Two-Port Element Example

**Electrical Inductor:**

- So, the inductor can be represented by the following lumped equivalent circuit:

- $L$ is the inductance given by $L = \frac{n^2}{R}$

- Stored energy in the inductor: $W_L = \frac{\phi^2}{2C} = \frac{\phi^2 R}{2}$

- Co-energy: $W_L^* = \frac{F_{MM}^2}{2R} = \frac{1}{2}LI^2$
Transducers

- **Definition:**
  A transducer is a device that converts one form of energy to another.

- **Broad Classification:**
  - Energy-Conserving: Electrostatic, Magnetic, etc.
  - Non-energy Conserving: Thermal, piezoresistive, etc.

- **Specific Classification:**
  - Linear versus nonlinear
  - Reciprocal versus anti-reciprocal
  - Direct versus indirect
Linear, Energy-conserving, Transducers:

- **Linear:**
  - Linearization about a mean may be required
  - Necessary for high-fidelity transduction of time-resolved info.

- **Energy-conserving:** (ref. Hunt, “Electroacoustics”)
  - Electromechanical coupling methods can be broadly classified according to whether the mechanical forces are produced under the action of electric fields on electric charges or by the interaction of magnetic fields and electric currents.

- There are five major electromechanical transducers:
1) **Electrodynamic**: motor/generator action are produced by the current, or the motion of an electric conductor located in a fixed transverse magnetic field (i.e., voice coil, solenoid, etc.).

2) **Electrostatic**: motor/generator action are produced by variations of the mechanical stress by maintaining a potential difference between two or more electrodes, one of which moves (i.e., condensor microphone, etc.).

3) **Magnetic**: motor/generator action are produced by variations of the tractive force tending to close the air gap in a ferromagnetic circuit.

4) **Piezoelectric**: motor/generator action are produced by the direct and converse piezoelectric effect - dielectric polarization gives rise to elastic strain and vice versa (i.e., tweeters, etc.).

5) **Magnetostrictive**: motor/generator action are produced by the direct and converse magnetostriction effect - magnetization of magnetostrictive materials gives rise to elastic strain and vice versa.
General Two-Port Theory for L.C. Transducers:

- Two-port networks expressed in either the impedance form or the admittance form

**Z-representation:**

\[
V = Z_{EB}I + T_{EM}U
\]

\[
F = T_{ME}I + Z_{MO}U
\]

or

\[
\begin{bmatrix}
V \\
F
\end{bmatrix} =
\begin{bmatrix}
Z_{EB} & T_{EM} \\
T_{ME} & Z_{MO}
\end{bmatrix}
\begin{bmatrix}
I \\
U
\end{bmatrix}
\]
Linear, Conservative Transducers

\[ Z_{EB} \equiv \frac{V}{I} \bigg|_{U=0} \rightarrow \text{blocked electrical impedance} \]

“Blocked”: no mechanical motion

\[ Z_{EF} \equiv \frac{V}{I} \bigg|_{F=0} \rightarrow \text{free electrical impedance} \]

“Free”: no force

\[ Z_{MO} \equiv \frac{F}{U} \bigg|_{I=0} \rightarrow \text{open-circuit mechanical impedance} \]

\[ Z_{MS} \equiv \frac{F}{U} \bigg|_{V=0} \rightarrow \text{short-circuit mechanical impedance} \]

\[ T_{EM} \equiv \frac{V}{U} \bigg|_{I=0} \rightarrow \text{open-circuit electromechanical transduction impedance} \]

\[ T_{ME} \equiv \frac{F}{I} \bigg|_{U=0} \rightarrow \text{blocked mechanical-electro transduction impedance} \]
• **Linear Reciprocal Transducers:**
  If $T_{EM} = T_{ME}$, then the transducer displays electromechanical reciprocity (i.e., electrostatic, crystal, or ceramic as reciprocal transducers)

  - We can define **Impedance Transformation Factor**
    \[
    \phi = \frac{T_{EM}}{T_{EB}}
    \]

  - Then we have
    \[
    \begin{cases}
    F = \phi Z_{EB} I + Z_{MO} U \\ 
    V = Z_{EB} I + \phi Z_{EB} U \\
    \end{cases}
    \]

    \[
    \begin{cases}
    F = \phi V + (Z_{MO} - \phi^2 Z_{EB}) U \\
    U = -\frac{1}{\phi} I + \frac{1}{\phi Z_{EB}} V \\
    \end{cases}
    \]

    Cannot directly write in a transformer or gyrator format!
Linear, Conservative Transducers

- But we can use a combination of a transformer and other components:

\[
\begin{align*}
F &= F_1 + Z_{MS} U \\
U &= -\frac{1}{\phi} I_1 = -\frac{1}{\phi} \left( I - \frac{V}{Z_{EB}} \right)
\end{align*}
\]

where

\[
\begin{bmatrix} F_1 \\ U \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ 0 & -1/\phi \end{bmatrix} \begin{bmatrix} V \\ I_1 \end{bmatrix}
\]

- \(Z_{MS}\): Short-circuit mechanical impedance

\[
Z_{ms} = Z_{MO} - \phi^2 Z_{EB} = Z_{MO} \left( 1 - k_e^2 \right)
\]

where

\[
k_e^2 = \frac{T_{EM}^2}{Z_{EB} Z_{MO}}
\]

- \(k_e\): Electromechanical coupling constant
Linear, Conservative, Transducers

General Two-Port Theory for L.C. Transducers:

- **Y-representation**:

\[
\begin{align*}
I &= Y_{EF} V + G_{EM} F \\
U &= G_{ME} V + Y_{MS} F
\end{align*}
\]

or

\[
\begin{bmatrix}
I \\
U
\end{bmatrix} = 
\begin{bmatrix}
Y_{EF} & G_{EM} \\
G_{ME} & Y_{MS}
\end{bmatrix}
\begin{bmatrix}
V \\
F
\end{bmatrix}
\]
Linear, Conservative, Transducers

\[ Y_{EB} \equiv \frac{I}{V} \bigg|_{U=0} \rightarrow \text{blocked electrical admittance} \]

\[ Y_{EF} \equiv \frac{I}{V} \bigg|_{F=0} \rightarrow \text{free electrical admittance} \]

\[ Y_{MO} \equiv \frac{U}{F} \bigg|_{I=0} \rightarrow \text{open circuit admittance} \]

\[ Y_{MS} \equiv \frac{U}{F} \bigg|_{V=0} \rightarrow \text{short circuit admittance} \]

\[ G_{EM} \equiv \frac{I}{F} \bigg|_{V=0} \rightarrow \text{short circuit electro-mechanical transduction admittance} \]

\[ G_{ME} \equiv \frac{U}{V} \bigg|_{F=0} \rightarrow \text{free mechanical-electro transduction admittance} \]
Reciprocal L.C. Transducers:

- **Y-representation**: If $G_{EM} = G_{ME} = G$, then the transducer displays electromechanical reciprocity.

Define the impedance transformation factor: $\phi'$, where $\phi' = \frac{-G}{Y_{EF}}$.

Now the canonical equations become

\[
\begin{align*}
I &= Y_{EF}V - \phi'Y_{EF}F \\
U &= -\phi'Y_{EF}V + Y_{MS}F
\end{align*}
\]

or

\[
\begin{bmatrix} I \\ U \end{bmatrix} = \begin{bmatrix} Y_{EF} & -\phi'Y_{EF} \\ -\phi'Y_{EF} & Y_{MS} \end{bmatrix} \begin{bmatrix} V \\ F \end{bmatrix}
\]
Circuit Representations:

- Y-representations:

Note that

\[ Y_{MO} = Y_{MS} - \frac{G^2}{Y_{EF}} \]
\[ Y_{EB} = Y_{EF} - \frac{G^2}{Y_{MS}} \]
Reciprocal and Anti-reciprocal

**General Circuit Representations:**

(a) **Reciprocal transducer** \( (T_{EM} = T_{ME} = T) \)

\[
V = Z_{EB} I + T_{EM} u = Z_{EB} I + T u
\]

\[
F = T_{EM} I + Z_{MO} u = TI + Z_{MO} u
\]

\[
V = Z_{EB} I + \phi Z_{EB} u
\]

\[
F = \phi Z_{EB} I + Z_{MO} u \quad \text{where} \quad \phi = \frac{T}{Z_{EB}}
\]

(b) **Anti-reciprocal transducer** \( (T_{EM} = -T_{ME}) \)

\[
V = Z_{EB} I + \phi_{M} u
\]

\[
F = -\phi_{M} I + Z_{MO} u
\]

\[
\downarrow
\]

\[
V = Z_{EF} I + \phi_{M} Y_{mo} F
\]

\[
u = \phi_{M} Y_{mo} I + Y_{mo} F
\]