Lumped-Element Modeling

- Last lecture
  - Conjugate power variables
  - Equivalent circuit modeling
  - One-port elements
    - Ideal Flow Source
    - Ideal Effort Source
    - Generalized Resistor

- Today:
  - One-port elements
    - Generalized Capacitor
    - Generalized Inertance
    - Co-energy
  - Kirchhoff’s Laws
  - Example
  - Laplace Transform

- Reading: Senturia, pp. 110-118.
Generalized Capacitor

- **Ideal Capacitor, or Compliance**: “C”
  - Stores potential energy associated with a displacement.
    
    \[ e = \Phi(q) \text{ or } q = \Phi^{-1}(e) \]
  - When a compliance has a non-zero effort (non-zero displacement), it is storing potential energy.

\[
W(q_1) = \int_{q_0}^{q_1} e \, dq = \int_{q_0}^{q_1} \Phi(q) \, dq \equiv \text{stored POTENTIAL ENERGY}\]

\[
W^*(e_1) = \int_{e_0}^{e_1} q \, de = \int_{e_0}^{e_1} \Phi^{-1}(e) \, de \equiv \text{stored POTENTIAL CO-ENERGY}\]
Generalized Capacitor

- **Ideal Compliance:** “C”

\[ e = \frac{1}{C} \int f \, dt \text{ or } e(j\omega) = \frac{1}{j\omega C} f(j\omega) \]

\[ e(s) = Z_C(s) f(s) \quad Z_C(s) = \frac{1}{sC} \]


Complex Impedance

\[ W(q_1) = \int_{0}^{q_1} \Phi(q) \, dq \]

\[ W^*(e_1) = \int_{0}^{e_1} \Phi^{-1}(e) \, de \]

If \( \Phi \) is linear,
then \( W(q_1) = W^*(e_1) \)

Ref. Senturia, p. 110.
**Example: Capacitance**

“electrical energy stored via charge”

\[ Q = CV \quad \text{and} \quad C = \frac{\varepsilon A}{g}, \] where \( \varepsilon \equiv \) dielectric permittivity

\[ W(Q) = \int e\,dq = \int V\,dQ = \int \frac{Q}{C}\,dQ = \frac{Q^2}{2C} = \frac{(CV)^2}{2C} = \frac{CV^2}{2} \]

\[ W^*(V) = \int q\,de = \int Q\,dV = \int CV\,dV = \frac{CV^2}{2} \]

Ref. Senturia, p. 111.
**Example:** Mechanical Spring

“strain energy stored via displacement”

\[ x = \frac{1}{k} F \] and \( C_m = \frac{1}{k} \), for a linearly elastic material.

\[ W(x) = \int e \, dq = \int F \, dx = \int kx \, dx = \frac{1}{2} kx^2 \]

\[ W^*(F) = \int q \, de = \int x \, dF = \int \frac{F}{k} \, dF = \frac{F^2}{2k} = \frac{(kx)^2}{2k} = \frac{1}{2} kx^2 \]

Ref. Senturia, p. 112.
Generalized Capacitor

- **Summary:**
  - **IDEAL:** no KE storage, dissipation, or source.
    - also must go through origin
    - conservative element
    - impedance analogy: \( e = \frac{1}{C} \int f \, dt \) or \( e(j\omega) = \frac{1}{j\omega C} f(j\omega) \)
  - Stores potential energy associated with a displacement.
    - Non-zero effort, non-zero displacement, PE is stored.
  - For **linear compliant systems** (linear dielectrics, Hooke’s Law, etc.)
    \[ W(q) = W^*(e) \]
Generalized Inertance

- **Ideal Inertance**: “\( I \)
  - Stores kinetic energy associated with momentum.
    \[
    f = \Psi(p) \quad \text{or} \quad p = \Psi^{-1}(f)
    \]
  - When a compliance has a non-zero flow (non-zero momentum), it is storing kinetic energy.

\[
W(p_1) = \int_0^{p_1} f\, dp = \int_0^{p_1} \Psi(p)\, dp \equiv \text{stored KINETIC ENERGY}
\]

\[
W^*(f_1) = \int_0^{f_1} p\, df = \int_0^{f_1} \Psi^{-1}(f)\, df \equiv \text{stored KINETIC CO-ENERGY}
\]
**Generalized Inertance**

- **Ideal Inertance:** “I”

\[
e = I \frac{df}{dt} \quad \text{or} \quad e(j\omega) = j\omega I f(j\omega)
\]

\[
e(s) = Z_L(s)f(s) \quad Z_C(s) = sL
\]

Complex Impedance

Ref. Senturia, p. 112.

\[
W(p_1) = \int_0^{p_1} \Psi(p) \, dp
\]

\[
W^*(f_1) = \int_0^{f_1} \Psi^{-1}(f) \, df
\]

If \( \Psi \) is linear then,

\[
W(p_1) = W^*(f_1)
\]
Example: Translating mass
“kinetic energy stored via momentum”

\[ I = M \]

\[ F = M \frac{du}{dt} \text{ and } p = Mu; \ "Newton's second law". \]

\[ W(p) = \int f \, dp = \int \frac{p}{M} \, dp = \frac{p^2}{2M} = \frac{1}{2} \left( \frac{Mu}{M} \right)^2 = \frac{1}{2} Mu^2 \]

\[ W^*(f) = \int p \, df = \int Mu \, du = \frac{1}{2} Mu^2 \]
Generalized Inertance

**Summary:**

- **IDEAL:** no PE storage, dissipation, or source.
  - also must go through origin
  - conservative element
  - impedance analogy: \( e = I \frac{df}{dt} \) or \( e(j\omega) = j\omega I f(j\omega) \)

- Stores kinetic energy associated with momentum.
  - Non-zero flow, non-zero momentum, KE is stored.

- For linear inertance systems (robots are a good counter example)
  \[ W(p) = W^*(f) \]
Generalized Kirchhoff’s Laws

KVL (or “KEL”): The oriented sum of all of the efforts around any closed path is zero.

\[ V_0 - V_1 - V_2 + V_3 = 0 \]

KCL (or “KFL”): The sum of all of the flows entering a node is zero.

\[ I_1 - I_2 - I_3 = 0 \]
For the IMPEDANCE analogy:

- Elements that share common FLOW and DISPLACEMENT are connected in SERIES.

- Elements that share common EFFORT are connected in PARALLEL.

For the admittance analogy, take the dual of the circuit.
Example

■ **Mass-spring-damper system:**

```
\[ \begin{align*}
\text{k} & \quad \xrightarrow{\text{x}} \quad \text{m} \\
\text{b} & \quad \text{F}
\end{align*} \]
```

“shared common displacement”, therefore connect in series.

```
\[ \begin{align*}
\text{v} & \quad \xrightarrow{\text{F}} \quad \frac{1}{\text{k}} \quad \text{m} \\
\text{b} &
\end{align*} \]
```

“single-path system”, therefore use **KEL**.

**MUST** use proper sign convention!

Example

**Mass-spring-damper system**

\[
\sum e_i = 0 = -F + e_k + e_m + e_m \quad \text{or} \\
F = M \frac{dv}{dt} + bv + k \int v \, dt \quad \text{or} \\
F = Ma + bv + kx \quad \text{or} \quad F = m\ddot{x} + b\dot{x} + kx
\]

\[
F(s) = \left(\frac{k}{s} + ms + b\right)v(s)
\]

\[
\frac{v(s)}{F(s)} = \frac{1}{\frac{k}{s} + ms + b} = \frac{s}{ms^2 + bs + k}
\]

- One zero, two poles

\[
s_1, s_2 = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}
\]
Laplace Transform

\[ F(s) = \int_{0}^{\infty} f(t)e^{-st} dt \]

**TABLE 13.1
AN ABBREVIATED LIST OF LAPLACE TRANSFORM PAIRS**

<table>
<thead>
<tr>
<th>( f(t) (t &gt; 0^-) )</th>
<th>TYPE</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta(t) )</td>
<td>(impulse)</td>
<td>1</td>
</tr>
<tr>
<td>( u(t) )</td>
<td>(step)</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>( t )</td>
<td>(ramp)</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>(exponential)</td>
<td>( \frac{1}{s + a} )</td>
</tr>
<tr>
<td>( \sin \omega t )</td>
<td>(sine)</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( \cos \omega t )</td>
<td>(cosine)</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( te^{-at} )</td>
<td>(damped ramp)</td>
<td>( \frac{1}{(s + a)^2} )</td>
</tr>
<tr>
<td>( e^{-at} \sin \omega t )</td>
<td>(damped sine)</td>
<td>( \frac{\omega}{(s + a)^2 + \omega^2} )</td>
</tr>
<tr>
<td>( e^{-at} \cos \omega t )</td>
<td>(damped cosine)</td>
<td>( \frac{s + a}{(s + a)^2 + \omega^2} )</td>
</tr>
</tbody>
</table>

### Laplace Transform


#### TABLE 13.2

**An Abbreviated List of Operational Transforms**

<table>
<thead>
<tr>
<th>OPERATION</th>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication by a constant</td>
<td>$Kf(t)$</td>
<td>$KF(s)$</td>
</tr>
<tr>
<td>Addition/subtraction</td>
<td>$f_1(t) + f_2(t) - f_3(t) + \cdots$</td>
<td>$F_1(s) + F_2(s) - F_3(s) + \cdots$</td>
</tr>
<tr>
<td>First derivative (time)</td>
<td>$\frac{df(t)}{dt}$</td>
<td>$sF(s) - f(0^-)$</td>
</tr>
<tr>
<td>Second derivative (time)</td>
<td>$\frac{d^2f(t)}{dt^2}$</td>
<td>$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$</td>
</tr>
<tr>
<td>nth derivative (time)</td>
<td>$\frac{d^nf(t)}{dt^n}$</td>
<td>$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$</td>
</tr>
<tr>
<td>Time integral</td>
<td>$\int_0^tf(x)dx$</td>
<td>$\frac{F(s)}{s}$</td>
</tr>
<tr>
<td>Translation in time</td>
<td>$f(t-a)u(t-a), a &gt; 0$</td>
<td>$e^{-as}F(s)$</td>
</tr>
<tr>
<td>Translation in frequency</td>
<td>$e^{-at}f(t)$</td>
<td>$F(s+a)$</td>
</tr>
<tr>
<td>Scale changing</td>
<td>$f(at), a &gt; 0$</td>
<td>$\frac{1}{a}F\left(\frac{s}{a}\right)$</td>
</tr>
<tr>
<td>First derivative (s)</td>
<td>$tf(t)$</td>
<td>$\frac{dF(s)}{ds}$</td>
</tr>
<tr>
<td>nth derivative (s)</td>
<td>$t^nf(t)$</td>
<td>$(-1)^n\frac{d^nF(s)}{ds^n}$</td>
</tr>
<tr>
<td>s integral</td>
<td>$\int_0^\infty \frac{f(t)}{t}du$</td>
<td>$\frac{1}{s}F(s)$</td>
</tr>
</tbody>
</table>
Laplace Transform

**Initial-value theorem:**  
\[ \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s) \]

**Final-value theorem:**  
\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \]