Structures

- Agenda:
  - Beams
    - Bending
    - Examples

- Reading: Senturia, Ch. 9, pp.211-219.
**Last lecture**

- Axial loading
  - Spring constant \( k_{axial} = \frac{EWH}{L} \)

- Bending
  - Reaction force \( F_R = F \)
  - Reaction moment \( M_R = FL \)
  - Shear force \( V = F \)
  - Internal bending moment \( M(x) = -F(L - x) \)
  - For distributed loading
    - \( q = -\frac{dV}{dx} \)
    - \( V = \frac{dW}{dx} \)
Bending of Beams

- **Stress in a beam with pure bending:**
  - Assume small deflections, i.e., “no strain in neutral axis”

![Diagram showing bending of beam, tension, neutral axis, and compression](image)

Ref. Senturia, pg 211.

The length of the differential element, \( dL = (\rho - z)d\theta \) where, \( \rho = \) the radius of curvature. At the neutral axis \( (z = 0) \), \( dx = \rho d\theta \), therefore, \( dL = dx - \frac{z}{\rho} \) dx.

So the corresponding stresses and strains can be expressed as

\[
\varepsilon_x = -\frac{z}{\rho} \quad \text{and} \quad \sigma_x = -\frac{zE}{\rho}
\]
**Bending of Beams**

Axial strain: \[ \varepsilon_x = -\frac{z}{\rho} \]

Axial stress: \[ \sigma_x = -\frac{zE}{\rho} \]

\[
M = -\int_{-H/2}^{H/2} z \sigma_x W dz = -\int_{-H/2}^{H/2} \frac{z^2 E}{\rho} W dz = -\left(\frac{1}{12} WH^3\right) \frac{E}{\rho}
\]

Therefore, \[ \frac{1}{\rho} = -\frac{M}{EI} \]

\( \rho \): curvature of radius
\( M \): bending moment
\( I \): Moment of inertia of the cross-section
Bending of Beams

**Centroid**
Center of gravity
First moment of an area: \( M = Ad \), \( A \) is the area, and \( d \) is the distance from the centroid of the area to the reference axis.
- Choose a reference axis
- Calculate the moment of each part
- Find the centroidal axis

\[
\sum A_i d_i = \left( \sum A_i \right) x
\]

**Bending Moment of Inertia:**

\[
I = \int_A y^2 \, dA \quad [m^4]
\]

**Example**

\[
I_z = wh^3 / 12 \quad I_z = \pi r^4 / 4
\]
DE for Beam Bending

- **Approximation for radius of curvature:**

![Diagram of beam bending with labels: x, dx, w(x), ds, θ(x)]


An increment of beam length $dx$ is related to $ds$ via

$$\cos(\theta) = \frac{dx}{ds}, \text{ for small } \theta \to dx \approx ds.$$  

The slope of the beam at any point is given by

$$\frac{dw}{dx} = \tan(\theta), \text{ for small } \theta \to \theta \approx \frac{dw}{dx}.$$  

For a given radius of curvature, $ds$ is related to $d\theta$ via

$$ds = \rho d\theta, \text{ so for small } \theta \to \frac{d\theta}{dx} \approx \frac{1}{\rho} \approx \frac{d^2w}{dx^2}.$$
DE for Beam Bending

- **Basic Differential Equations for Beam Bending:**

  \[ \frac{d^2 w}{dx^2} = \frac{1}{\rho}. \]
  
  Now that we have a relationship between \( w(x) \) and \( \rho \), we can express the moment and shear forces as a function of \( w(x) \),

  **Moments:**
  \[ \frac{d^2 w}{dx^2} = -\frac{M}{EI}, \]
  
  now recall \( V = \frac{dM}{dx} \)

  **Shear:**
  \[ \frac{d^3 w}{dx^3} = -\frac{V}{EI}, \]
  
  now recall \( -q = \frac{dV}{dx} \)

  **Uniform Load:**
  \[ \frac{d^4 w}{dx^4} = \frac{q}{EI}, \]
Cantilever Beam

- **Cantilever Beam with Point Load:**

Recall from last lecture, we found that

\[ M(x) = -F(L-x), \]

so the governing DE is

\[ \frac{d^2w}{dx^2} = -\frac{M}{EI} = \frac{F}{EI} (L-x), \]

which is 2nd order ODE.

Integrating the above equation twice, we have

\[ w(x) = A + Bx + \frac{FL}{2EI} x^2 - \frac{F}{6EI} x^3 \]

Using the boundary conditions, we obtain the beam deflection equation:

\[ w(x) = \frac{FLx^2}{2EI} \left( 1 - \frac{x}{3L} \right) \]

Boundary conditions:

\[ w(0) = 0 \quad \frac{dw}{dx} \bigg|_{x=0} = 0 \]

Maximum deflection:

\[ w(x) = \frac{FL^3}{3EI} \]

Spring constant:

\[ k = \frac{3EI}{L^3} = \frac{EWH^3}{4L^3} \]
**Cantilever Beam**

**Maximum Stress:**

Recall from last lecture, we found that \( \varepsilon_x = -\frac{z}{\rho} \) and \( \sigma_x = -\frac{zE}{\rho} \),

but \( \frac{1}{\rho} = \frac{d^2w}{dx^2} = \frac{F}{EI} (L - x) \).

\[ \sigma_x = -\frac{zEF}{EI} (L - x) \], the maximum magnitude occurs at \( z = \left\lfloor \frac{H}{2} \right\rfloor \) (the top and bottom surface) and \( x = 0 \rightarrow \left\| \sigma_{\text{max}} \right\| = \frac{HLF}{2I} = \frac{6LF}{WH^2} \).

Clearly, there were be a maximum load that this cantilever can withstand without breaking.
Cantilever Beam

- Linear beam theory limit


- Small deflection approximation is good for $y < 0.3L$
Anticlastic Curvature

- Transverse Strain via Poisson effect:

Recall, $\varepsilon_x = -\frac{z}{\rho}$, therefore, $\varepsilon_y = -\nu \varepsilon_x = \frac{\nu z}{\rho}$

Usually, this effect is too small to see for slender beams because $\rho$ tend to be very large with respect to the $W$ or $H$.

Simple Beam Equations

Ref. CMU 18-819 course notes (G. Fedder)