Structures

- **Agenda:**
  - Beams
    - Axially Loaded
    - Bending

- Reading: Senturia, Ch. 9, pp. 200-215.
Microsystems are comprised of many different types of structural elements.

Stiffness? Resonance frequency?

Tethers for parallel plate capacitors

Ref. Young, et al., HH98

Tethers for accelerometers

Ref. ADI

Distributed loading

Bending

Twisting

Axial loading
Axially Loaded Beams

The uniform axial stress is \( \sigma = \frac{F}{WH} \). If the beam is comprised of a linearly elastic material, then the strain is \( \varepsilon = \frac{\sigma}{E} = \frac{F}{EWH} \) (Hooke's Law). The corresponding change in the length of the beam is \( \delta L = \varepsilon L = \frac{FL}{EWH} \). The spring constant for this axially loaded beam can be determined by \( F = k\delta L \), so \( k = \frac{EWH}{L} \).
Axially Loaded Beams

- **Axially Loaded Beams of Varying Cross-section**

If the beam has a non-uniform cross-section \( A=A(x) \), we can slice the beam into differential elements of length \( dx \). Therefore, we can find the change in length for each individual element by \( \delta(dx) = \frac{Fd(x)}{EA(x)} \). We can obtain the total change in length for the entire beam by integrating, \( \delta L = \int_{0}^{L} \delta(dx) dx = \int_{0}^{L} \frac{Fd(x)}{EA(x)} \). The spring constant for the non-uniform, axially loaded beam can be determined by

\[
k = \frac{F}{\delta L} = \left[ \int_{0}^{L} \frac{dx}{EA(x)} \right]^{-1}.
\]
Examples: in-plane stress generated via thermal mismatch from packaging, film growth, film deposition, etc.

\[ \varepsilon = \frac{1}{2} \Delta \]

\[ \varepsilon / 2 \]

Reaction force

Thermal strain without clamping boundaries:

\[ \varepsilon_{thermal} = \alpha_T \cdot \Delta T \]

\( \alpha: \) Coefficient of thermal expansion

Total strain must be zero due to the clamped-clamped boundaries

\[ \varepsilon_{thermal} + \frac{\sigma_{thermal}}{E} = 0 \]

\( \sigma_{thermal}: \) thermal induced stress

\[ \sigma_{thermal} = -E \cdot \alpha_T \cdot \Delta T \]

Compressive thermal stress

Ref. Senturia, p. 204.
Stresses on Inclined Sections

\[ F_N = F \cos \theta \]
\[ F_V = F \sin \theta \]

\[ \sigma(\theta) = \frac{F}{A} \cos^2 \theta \]
\[ \tau(\theta) = \frac{F}{A} \cos \theta \sin \theta = \frac{F}{2A} \sin 2\theta \]

Shear stress is highest at +/-45°

**Bending of Beams**

- **Beams:** The “beam assumption” requires $W << L$ and $H << L$.

- **Types of Support:**
  - **fixed:** zero deflection and zero slope
  - **free:** no constraints
  - **pinned or simply supported:** zero deflection, variable slope
  - **pinned on rollers:** zero vertical deflection, movable horizontally, variable slope

Ref. Senturia, pg 207.
Bending of Beams

- **Types of Loads:**
  - **Point Load:** examples: comb-drives, tethered floating elements, etc.
    - Note: the load is the total force $F$, therefore the units will be Newton
    - If the beam width is considered, the load may be specified as the force per unit width, $F'$, with unit of N/m. So, $F = F' \times W$.
  - **Distributed Load:** examples: pressure sensors, microphones, etc.
    - Note: the load is given by the force per unit length, $q$, with unit of N/m
    - If beam width is considered, the load is equivalent to a pressure, $P$, with unit of N/m$^2$. So, $q = P \times W$.

Bending of Beams

- Loads, Shear Forces, Moments: Sign Conventions

Reaction Forces and Moments:

The beam that we have shown cannot be in equilibrium, unless we include reaction forces at the supports. Consider the cantilever beam...

For equilibrium, $\sum F = 0$ and $\sum M = 0$. Looking at the entire beam, we see that,

$\sum F = 0 = F - F_R = 0$, therefore $F_R = F$.

$\sum M_0 = 0 = -M_R + FL = 0$, therefore $M_R = FL$ (negative direction).
Bending of Beams

- **Shear Forces and Moments:**
  (at any point in the beam)

\[
\sum F = 0 \quad \text{and} \quad \sum M = 0.
\]

Looking at a slice at \( x \) we find:

\[
\sum F = 0 = -F + V(x) = 0, \quad \text{therefore} \quad V = F.
\]

\[
\sum M_L = 0 = -M(x) + F(L - x) = 0, \quad \text{therefore} \quad M(x) = -F(L - x).
\]
Bending of Beams

**Equilibrium of a fully loaded differential element:**

For equilibrium, \( \sum F=0 \) and \( \sum M=0 \) (w.r.t. left hand edge).

Looking at the element \( dx \), we see that,

\[
\sum F=0 = q \, dx + (V + dV) - V = 0, \text{ therefore } - q = \frac{(V+dV)-V}{dx} = \frac{dV}{dx}.
\]

\[
\sum M=0 = (M + dM) - M - (V + dV)dx - q \, dx \, \frac{dx}{2} = 0,
\]

therefore \( V = \frac{(M+dM)-M}{dx} = \frac{dM}{dx} \) (neglecting \( O(dx^2) \) terms).